Second order linear ODE

Standard form: $y'' + p(t)y' + q(t)y = g(t)$

Homogeneous: $g(t) = 0$

If you know the general solution to a homogeneous ODE, then by variation of parameters (will be seen later), you will be able to solve the corresponding non-homogeneous ODE

Existence & Uniqueness theorem: for the IVP

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

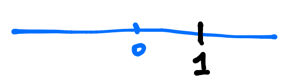
If $p(t), q(t), g(t)$ are continuous over an interval (a,b) that contain t_0 $(a < t_0 < b)$, then the IVP has a unique solution over (a,b).

Example: Find the interval of existence to the IVP

$$t^2y'' - 4ty' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 4.$$

Std. form: $y'' - \frac{4}{t}y' + \frac{4}{t^2} = 0$

blows up at $t=0$



the solution exists on $(0, \infty)$

Principle of Superposition

1) If y_1, y_2 are solutions to a 2^{nd}-order linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0.$$

then for ANY number c_1, c_2, the function $c_1y_1 + c_2y_2$ is also a solution

2) Moreover, if y_1, y_2 are linearly independent (meaning the Wronskian $W(y_1, y_2) \not\equiv 0$), then the general solution is $c_1y_1 + c_2y_2$ where c_1, c_2 are arbitrary constants.

Example: $y'' - 2y' - 3y = 0$.

1) $y_1 = e^{3t}$, $y_2 = e^{-t}$ are solutions.

$$y_1'' - 2y_1' - 3y_1 = 9e^{3t} - 2\times 3e^{3t} - 3e^{3t} = (9-6-3)e^{3t} = 0$$

$$y_2'' - 2y_2' - 3y_2 =$$

2) $\overset{c_1}{20}e^{3t} - \overset{c_2}{99}e^{-t}$ is also a solution.

$$20e^{3t} - 99e^{-t} = 20y_1 - 99y_2 \triangleq Y$$

$$Y'' - 2Y' - 3Y = (\overset{c_1}{20}y_1 - \overset{c_2}{99}y_2)'' - 2(\overset{c_1}{20}y_1 - \overset{c_2}{99}y_2)' - 3(\overset{c_1}{20}y_1 - \overset{c_2}{99}y_2)$$

$$= 20y_1'' - 99y_2'' - 2(20y_1') + 2(99y_2') - 3(20y_1) + 3(99y_2)$$

$$= (\overset{c_1}{20}y_1'' - 2\times\overset{c_1}{20}y_1' - 3\times\overset{c_1}{20}y_1) - (\overset{-c_2}{99}y_2'' - 2\times\overset{-c_2}{99}y_2' - 3\times\overset{-c_2}{99}y_2)$$

$$= 0$$

Wronskian of two functions

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Two functions y_1, y_2 are linearly dependent if $y_1 = k y_2$ for some number k. Equivalently, if the Wronskian $W(y_1, y_2)$ is constantly zero. Otherwise, we say y_1, y_2 are linearly independent.

Examples: 1) $y_1 = t^2$, $y_2 = t^3$

$$W(t^2, t^3) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = 3t^4 - 2t^4 = t^4 \not\equiv 0. \Rightarrow \text{ independent.}$$

2) $y_1 = e^{3t}$, $y_2 = 5e^{3t}$

$$W(e^{3t}, 5e^{3t}) = \begin{vmatrix} e^{3t} & 5e^{3t} \\ 3e^{3t} & 15e^{3t} \end{vmatrix} = 15e^{6t} - 15e^{6t} = 0$$

$\Rightarrow$ dependent.

3) $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$. $r_1 \neq r_2$

$$W(e^{r_1 t}, e^{r_2 t}) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = r_2 e^{(r_1+r_2)t} - r_1 e^{(r_1+r_2)t}$$

$$= (r_2 - r_1) e^{(r_1+r_2)t}$$

$r_1 \neq r_2$, $W \not\equiv 0 \Rightarrow$ independent

4) $y_1 = \sin^2 t$, $y_2 = \cos 2t - 1$

$$W(\sin^2 t, \cos 2t - 1) = \begin{vmatrix} \sin^2 t & \cos 2t - 1 \\ 2\sin t \cos t & -2\sin 2t \end{vmatrix}$$

$$= -2\sin^2 t \sin 2t - 2\sin t \cos t (\cos 2t - 1)$$

$\sin 2t = 2\sin t \cos t$

$$= \sin 2t \left(-2\sin^2 t + 1 - \cos 2t\right)$$

$\sin^2 t = \frac{1-\cos 2t}{2}$ $\quad = 0.$

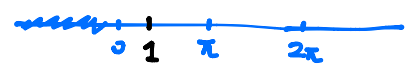
Attendence Quiz: HW 1b 2a

1. Find interval of existence to $y'' - \cot t\, y' + (\ln t) y = e^{5t}$ $\quad y(1) = 0, \ y'(1) = 2.$
2. Find if $y_1 = e^{2t}\sin t$, $y_2 = e^{2t}\cos t$ are linearly dependent.

1. $\cot t$ blows up when $\sin t = 0$ $\qquad \cot t = \frac{\cos t}{\sin t}$

   i.e. $t = k\pi$, $k = 0, \pm 1, \pm 2, \cdots$

   $\ln t$ is not defined when $t < 0$.



   IoE: $(0, \pi)$

2. $$W = \begin{vmatrix} e^{2t}\sin t & e^{2t}\cos t \\ 2e^{2t}\sin t + e^{2t}\cos t & 2e^{2t}\cos t + e^{2t}(-\sin t) \end{vmatrix}$$

$$= e^{2t}\sin t\left(2e^{2t}\cos t - e^{2t}\sin t\right) - e^{2t}\cos t\left(2e^{2t}\sin t + e^{2t}\cos t\right)$$

$\cos^2 t + \sin^2 t = 1$

$\cos^2 t - \sin^2 t = \cos 2t$

$$= \cancel{2e^{4t}\sin t\cos t} - e^{4t}\sin^2 t \cancel{- 2e^{4t}\cos t\sin t} - e^{4t}\cos^2 t$$

$$= e^{4t}\left(-\sin^2 t - \cos^2 t\right) = -1\, e^{4t}$$

Coming back to the ODE $y'' - 2y' - 3y = 0$.

Knowing that e^{3t}, e^{-t} are solutions to this homogeneous linear ODE.

From the above example. $W(e^{3t}, e^{-t}) \not\equiv 0$

The general solution is $y = C_1 e^{3t} + C_2 e^{-t}$.

Second order linear homogeneous ODE with constant coefficients.

$$ay''+by'+cy=0$$, where a, b, c are numbers.

Idea: Try $y=e^{rt}$. (derivatives are multiples of original function)

$$a(e^{rt})''+b(e^{rt})'+ce^{rt}=ar^2e^{rt}+bre^{rt}+ce^{rt}$$

$$=(ar^2+br+c)e^{rt}=0$$

This means, if $y=e^{rt}$ is a solution, then r must satisfy a quadratic equation

$$ar^2+br+c=0$$

called characteristic equation.

In case you obtain two distinct real roots, r_1, r_2, we know that $y_1=e^{r_1t}$, $y_2=e^{r_2t}$ are solutions, with $W(y_1,y_2)\neq 0$.
By the principle of superposition, the general solution is

$$y=C_1e^{r_1t}+C_2e^{r_2t}.$$

Examples: 1) $y''-2y'-3y=0$

Character eqn: $r^2-2r-3=0 \Rightarrow (r+1)(r-3)=0 \Rightarrow r=3$ or -1

General solution: $y = C_1 e^{3t} + C_2 e^{-t}$.

2) $y'' - 5y' + 6y = 0$

characteristic eqn: $r^2 - 5r + 6 = 0$, $(r-2)(r-3) = 0$

$r = 2$ or 3. gen. sol'n: $y = C_1 e^{3t} + C_2 e^{2t}$

3) $y'' - 5y' - 6y = 0$

char. eqn. $r^2 - 5r - 6 = 0$. $(r+1)(r-6) = 0$

$r = 6$ or -1. Gen. sol'n: $y = C_1 e^{6t} + C_2 e^{-t}$.

4) $2y'' + 7y' + 3y = 0$.

char. eqn. $2r^2 + 7r + 3 = 0$. $(2r+1)(r+3) = 0$ Criss-cross method.

$r = -\frac{1}{2}$ or -3. General solution: $y = C_1 e^{-\frac{1}{2}t} + C_2 e^{-3t}$

Example: $y'' - 4y' - 6y = 0$. $y(0) = 1$, $y'(0) = 0$

$r^2 - 4r - 6 = 0$. $r = \dfrac{4 \pm \sqrt{16 - 4(-6)}}{2} = \dfrac{4 \pm \sqrt{40}}{2} = 2 \pm \sqrt{10}$

Gen. sol'n: $y = C_1 e^{(2+\sqrt{10})t} + C_2 e^{(2-\sqrt{10})t}$

$y(0) = 1 \Rightarrow C_1 + C_2 = 1 \Rightarrow C_1 = 1 - C_2$

$y'(0) = 0 \Rightarrow C_1 \cdot (2 + \sqrt{10}) + C_2 (2 - \sqrt{10}) = 0$

$\Rightarrow (2+\sqrt{10}) - (2+\sqrt{10}) C_2 + (2 - \sqrt{10}) C_2 = 0$

$(2+\sqrt{10}) = 2\sqrt{10}\, C_2 \Rightarrow C_2 = \dfrac{2+\sqrt{10}}{2\sqrt{10}} = \dfrac{2\sqrt{10}+10}{20} = \dfrac{\sqrt{10}+5}{10}$

$$C_1 = 1 - C_2 = 1 - \frac{\sqrt{10}+5}{10} = \frac{-\sqrt{10}-5}{10}.$$

Solution to the IVP $y = \frac{-\sqrt{10}-5}{10} e^{(2+\sqrt{10})t} + \frac{\sqrt{10}+5}{10} e^{(2-\sqrt{10})t}$

In addition, determine the long term behavior of the solution.

$$\lim_{t\to\infty} y(t) = -\infty.$$

$e^{(2-\sqrt{10})t} \to 0$ as $t \to \infty$. y is controlled by the first term and the first term goes to $-\infty$.

Example: $y'' - 5y' + 6y = 0$, $y(0) = \alpha$. $y'(0) = 1$

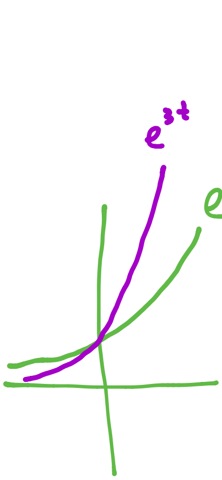
Determine the critical value of α when the long term behavior changes.

Gen. sol'n: $y = C_1 e^{2t} + C_2 e^{3t}$

$y(0) = \alpha \Rightarrow C_1 + C_2 = \alpha \Rightarrow C_1 = \alpha - C_2$

$y'(0) = 1 \Rightarrow 2C_1 + 3C_2 = 1 \Rightarrow 2(\alpha - C_2) + 3C_2 = 1$

$\Rightarrow C_2 = 1 - 2\alpha \Rightarrow C_1 = 3\alpha - 1$



$$y = (3\alpha - 1)e^{2t} + (1 - 2\alpha)e^{3t}$$

Since e^{3t} increases faster than e^{2t}, as t becomes large, the solution is dominated by $(1-2\alpha)e^{3t}$.

When $1-2\alpha>0$, $y\to+\infty$

$1-2\alpha<0$, $y\to-\infty$

So the behavior changes when $1-2\alpha=0. \Rightarrow \alpha=\frac{1}{2}$

Critical value $=\frac{1}{2}$

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